Voting

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Abstract

The problems with voting.
Voting
The Problem

Milk  Wine  Beer
Beer  Wine  Milk
Milk  Wine  Beer
Wine  Beer  Milk
Voting
The Problem

Milk
Wine
Beer

Beer
Wine
Milk

Milk
Wine
Beer

Wine
Beer
Milk
Voting
The Problem

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<td>Plurality</td>
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Milk, Wine, Beer
Voting

The Problem

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<tr>
<td>Runoff</td>
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Milk Wine Beer

Beer Wine Milk

Wine Beer Milk
Voting

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Milk
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Plurality
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Beer
Wine
Milk

Milk
Wine
Beer

Wine
Beer
Milk
Voting
Possible Solutions

Symmetry

- **Reflectional symmetry**: If one agent prefers A to B and another one prefers B to A then their votes should cancel each other out.

- **Rotational symmetry**: If one agent prefers A,B,C and another one prefers B,C,A and another one prefers C,A,B then their votes should cancel out.
Symmetry

- **Reflectional symmetry**: If one agent prefers A to B and another one prefers B to A then their votes should cancel each other out.

- **Rotational symmetry**: If one agent prefers A, B, C and another one prefers B, C, A and another one prefers C, A, B then their votes should cancel out.

- Plurality vote violates reflectional symmetry, so does runoff voting.

- Pairwise comparison violates rotational symmetry.
Borda Count

1. With $x$ candidates, each agent awards $x$ to points to his first choice, $x - 1$ points to his second choice, and so on.

2. The candidate with the most points wins.

Borda satisfies both reflectional and rotational symmetry.
There is a set of $A$ agents, and $O$ outcomes.

Each agent $i$ has a preference function $>_{i}$ over the set of outcomes.

Let $>^{*}$ be the global set of social preferences. That is, what we want the outcome to be.
Definition (Desirable Voting Outcome Conditions)

1. \( \succ^* \) exists for all possible inputs \( \succ_i \).
**Definition (Desirable Voting Outcome Conditions)**

1. \( >^* \) exists for all possible inputs \( >_i \);
2. \( >^* \) exists for every pair of outcomes.
Definition (Desirable Voting Outcome Conditions)

1. >* exists for all possible inputs >i;
2. >* exists for every pair of outcomes
3. >* is asymmetric and transitive over the set of outcomes
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1. \( >^* \) exists for all possible inputs \( >^i \);
2. \( >^* \) exists for every pair of outcomes
3. \( >^* \) is asymmetric and transitive over the set of outcomes
4. \( >^* \) should be Pareto efficient.
Definition (Desirable Voting Outcome Conditions)

1. >* exists for all possible inputs >i;
2. >* exists for every pair of outcomes
3. >* is asymmetric and transitive over the set of outcomes
4. >* should be Pareto efficient.
5. The scheme used to arrive at >* should be independent of irrelevant alternatives.
**Definition (Desirable Voting Outcome Conditions)**

1. $\succ^*$ exists for all possible inputs $\succ_i$;
2. $\succ^*$ exists for every pair of outcomes;
3. $\succ^*$ is asymmetric and transitive over the set of outcomes;
4. $\succ^*$ should be Pareto efficient.
5. The scheme used to arrive at $\succ^*$ should be independent of irrelevant alternatives.
6. No agent should be a dictator in the sense that $\succ^*$ is always the same as $\succ_i$, no matter what the other $\succ_j$ are.
Theorem (Arrow’s Impossibility)

There is no social choice rule that satisfies the six conditions.
Voting
Possible Solutions
We are Doomed

Kenneth Arrow

Theorem (Arrow’s Impossibility)

There is no social choice rule that satisfies the six conditions.

- Plurality voting relaxes 3 and 5. Adding a third candidate can wreak havoc.
- Pairwise relaxes 5.
- Borda violates 5.
Borda Example

1. $a > b > c > d$
2. $b > c > d > a$
3. $c > d > a > b$
4. $a > b > c > d$
5. $b > c > d > a$
6. $c > d > a > b$
7. $a > b > c > d$
Borda Example

1. $a > b > c$
2. $b > c > a$
3. $c > a > b$
4. $a > b > c$
5. $b > c > a$
6. $c > a > b$
7. $a > b > c$