# Voting

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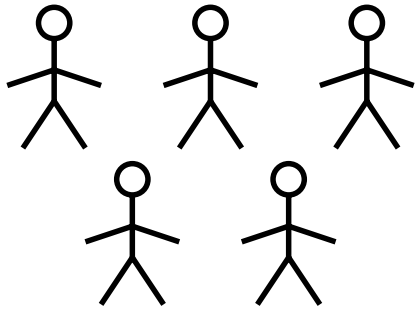
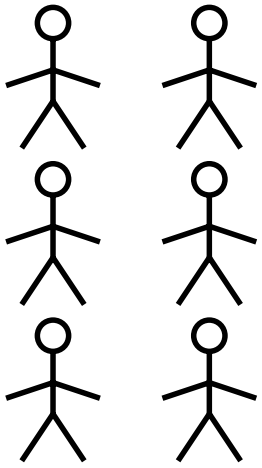
## Abstract

The problems with voting.



Beer
Wine
Milk



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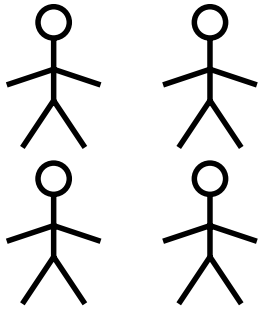

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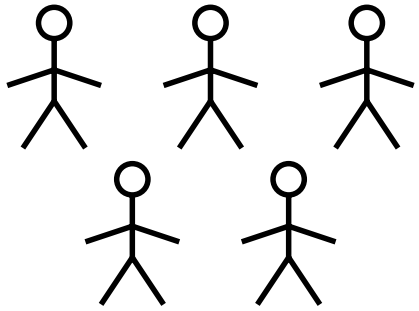

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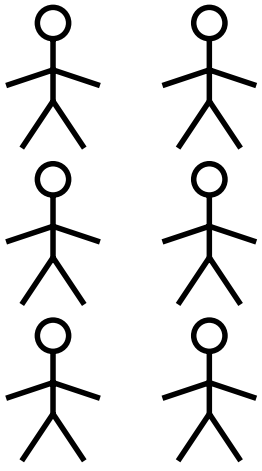

Milk
Wine
Beer

| | Beer | Wine | Milk |
|---|---|---|---|
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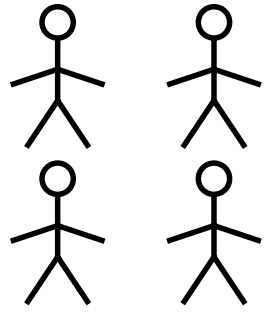

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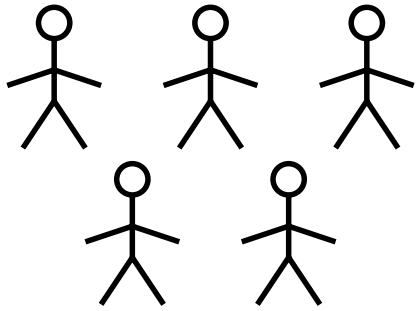

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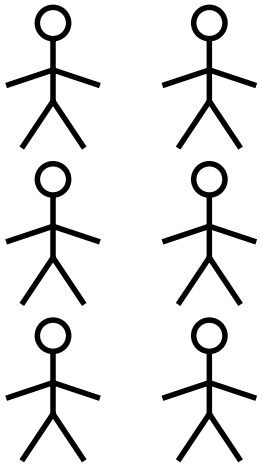
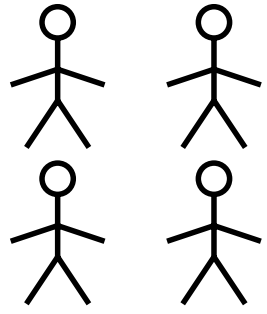


Milk
Wine
Beer

| | Beer | Wine | Milk |
|---|---|---|---|
| Plurality | | | |
| | | | |
| | | | |



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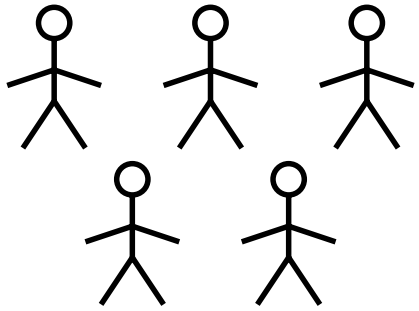

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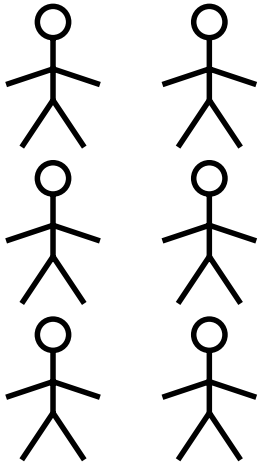
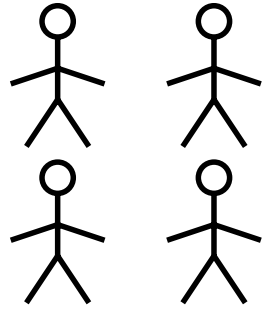


Milk
Wine
Beer

| | Beer | Wine | Milk |
|---|---|---|---|
| Plurality | 5 | 4 | 6 |
| | | | |
| | | | |



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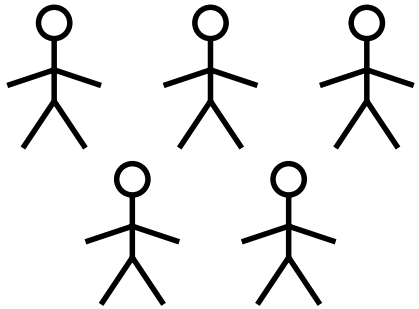

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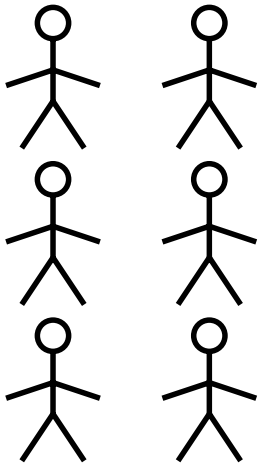
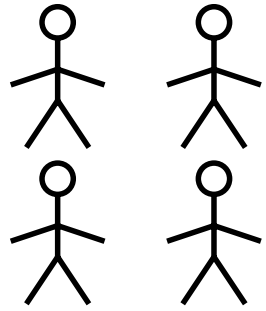


Milk
Wine
Beer

| | Beer | Wine | Milk |
|---|---|---|---|
| Plurality | 5 | 4 | 6 |
| Runoff | | | |
| | | | |



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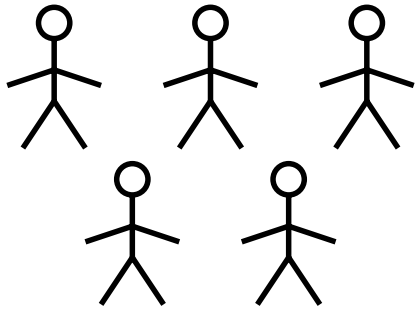

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Wine
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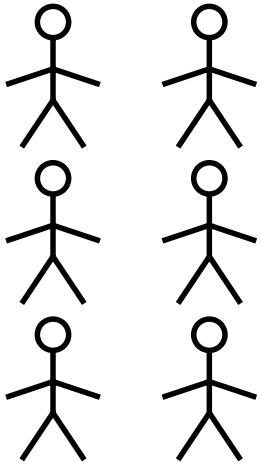
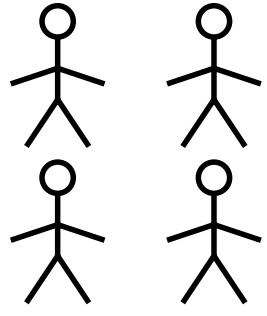


Milk
Wine
Beer

| | Beer | Wine | Milk |
|---|---|---|---|
| Plurality | 5 | 4 | 6 |
| Runoff | 5,9 | 4 | 6,6 |
| | | | |



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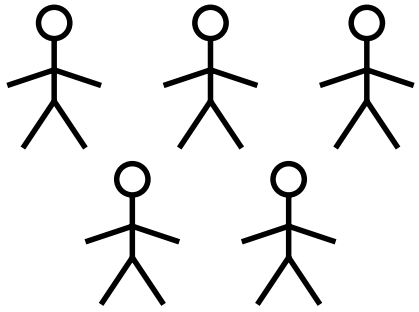

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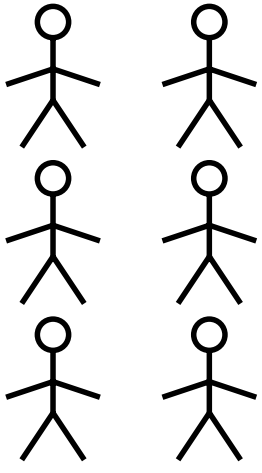

Milk
Wine
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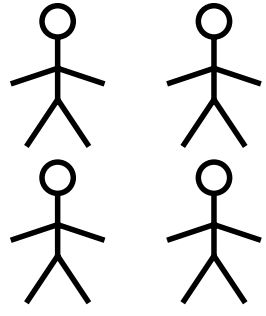
| | Beer | Wine | Milk |
|---|---|---|---|
| Plurality | 5 | 4 | 6 |
| Runoff | 5,9 | 4 | 6,6 |
| Pairwise | | | |



Wine
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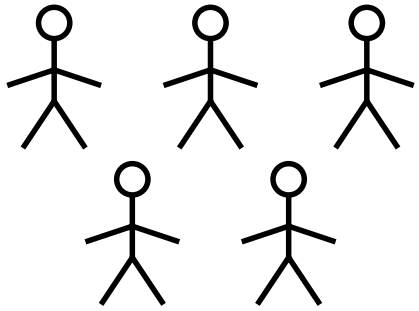

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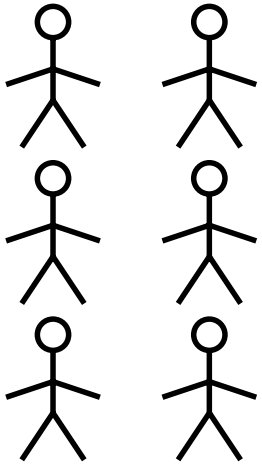
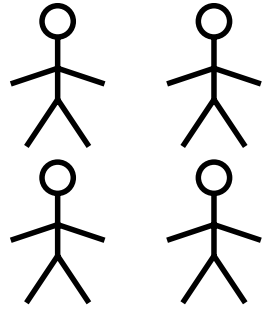


Milk
Wine
Beer

| | Beer | Wine | Milk |
|---|---|---|---|
| Plurality | 5 | 4 | 6 |
| Runoff | 5,9 | 4 | 6,6 |
| Pairwise | 1 | 2 | 0 |



Wine
Beer
Milk

# Symmetry

- Reflectional symmetry: If one agent prefers A to B and another one prefers B to A then their votes should cancel each other out.
- Rotational symmetry: If one agent prefers A,B,C and another one prefers B,C,A and another one prefers C,A,B then their votes should cancel out.

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- Rotational symmetry: If one agent prefers A,B,C and another one prefers B,C,A and another one prefers C,A,B then their votes should cancel out.

- Plurality vote violates reflectional symmetry, so does runoff voting.
- Pairwise comparison violates rotational symmetry.

# Borda Count


Jean-Charles de Borda. 1733–1799.

1. With $x$ candidates, each agent awards $x$ to points to his first choice, $x-1$ points to his second choice, and so on.
2. The candidate with the most points wins.

Borda satisfies both reflectional and rotational symmetry.

# Formalization

- There is a set of $A$ agents, and $O$ outcomes.
- Each agent $i$ has a preference function $>_i$ over the set of outcomes.
- Let $>^*$ be the global set of social preferences. That is, what we want the outcome to be.

## Definition (Desirable Voting Outcome Conditions)

1. $>^*$ exists for all possible inputs $>_i$

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5. The scheme used to arrive at $>^*$ should be independent of irrelevant alternatives.

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1. $>^*$ exists for all possible inputs $>_i$
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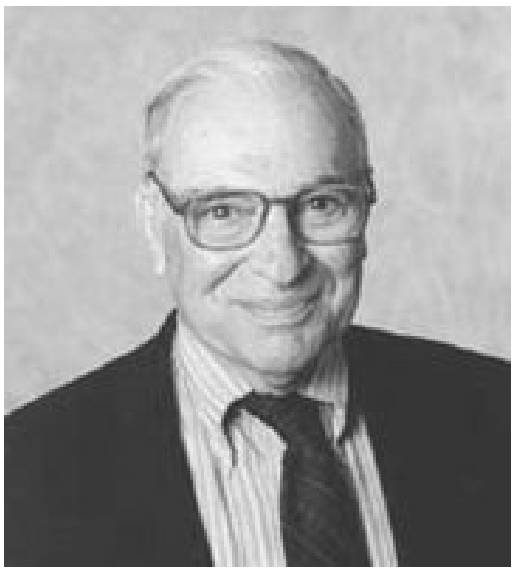
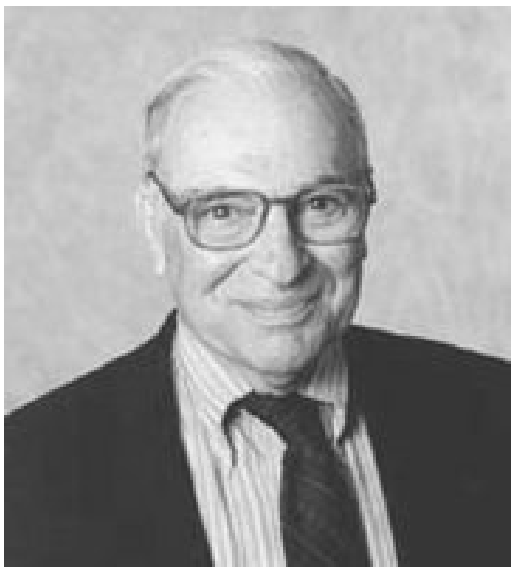
3. $>^*$ is asymmetric and transitive over the set of outcomes
4. $>^*$ should be Pareto efficient.
5. The scheme used to arrive at $>^*$ should be independent of irrelevant alternatives.
6. No agent should be a dictator in the sense that $>^*$ is always the same as $>_i$, no matter what the other $>_j$ are.



Kenneth Arrow

**Theorem (Arrow's Impossibility)**

*There is no social choice rule that satisfies the six conditions.*



Kenneth Arrow

**Theorem (Arrow's Impossibility)**

*There is no social choice rule that satisfies the six conditions.*

- Plurality voting relaxes 3 and 5. Adding a third candidate can wreak havoc.
- Pairwise relaxes 5.
- Borda violates 5.

# Borda Example

1. $a > b > c > d$
2. $b > c > d > a$
3. $c > d > a > b$
4. $a > b > c > d$
5. $b > c > d > a$
6. $c > d > a > b$
7. $a > b > c > d$

# Borda Example

1. $a > b > c$
2. $b > c > a$
3. $c > a > b$
4. $a > b > c$
5. $b > c > a$
6. $c > a > b$
7. $a > b > c$