Voting and Mechanism Design

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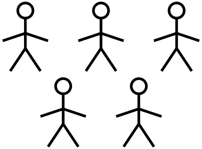
Abstract

Voting, Mechanism design, and distributed algorithmics mechanism design. Chapter 8.

Why Vote?

- Common way of aggregating agents' preferences.
- Well understood.
- But, centralized.

The Voting Problem



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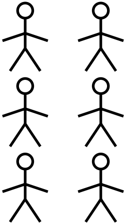

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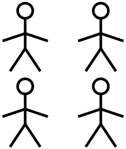

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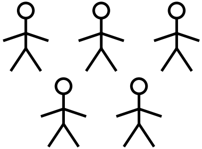
	Beer	Wine	Milk
Plurality			
Runoff			
Pairwise			

The Voting Problem



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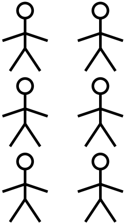

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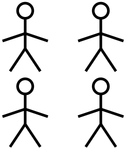

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	Beer	Wine	Milk
Plurality	5	4	6
Runoff			
Pairwise			

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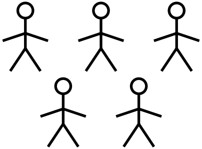

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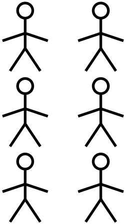

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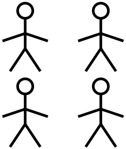

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	Beer	Wine	Milk
Plurality	5	4	6
Runoff	5,9	4	6,6
Pairwise			

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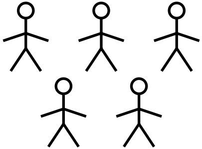

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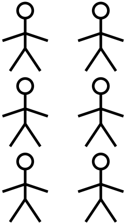

Milk
Wine
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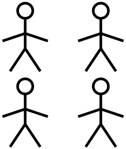
	Beer	Wine	Milk
Plurality	5	4	6
Runoff	5,9	4	6,6
Pairwise	1	2	0



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Symmetry

- Reflectional symmetry: If one agent prefers A to B and another one prefers B to A then their votes should cancel each other out.
- Rotational symmetry: If one agent prefers A,B,C and another one prefers B,C,A and another one prefers C,A,B then their votes should cancel out.

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- Plurality vote violates reflectional symmetry, so does runoff voting.
- Pairwise comparison violates rotational symmetry.

Borda Count

Jean-Charles de Borda. 1733–1799.

1. With x candidates, each agent awards x to points to his first choice, $x - 1$ points to his second choice, and so on.
2. The candidate with the most points wins.

Borda satisfies both reflectional and rotational symmetry.

Formalization

- There is a set of A agents, and O outcomes.
- Each agent i has a preference function $>_i$ over the set of outcomes.
- Let $>^*$ be the global set of social preferences. That is, what we want the outcome to be.

Definition (Desirable Voting Outcome Conditions)

1. $>^*$ exists for all possible inputs $>_i$

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5. The scheme used to arrive at $>^*$ should be independent of irrelevant alternatives.

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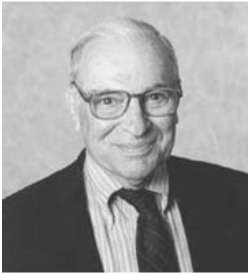
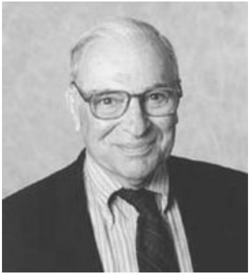
4. $>^*$ should be Pareto efficient.
5. The scheme used to arrive at $>^*$ should be independent of irrelevant alternatives.
6. No agent should be a dictator in the sense that $>^*$ is always the same as $>_i$, no matter what the other $>_j$ are.



Kenneth Arrow

Theorem (Arrow's Impossibility)

There is no social choice rule that satisfies the six conditions.



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There is no social choice rule that satisfies the six conditions.

- Plurality voting relaxes 3 and 5. Adding a third candidate can wreak havoc.
- Pairwise relaxes 5.
- Borda violates 5.

Borda Example

1. $a > b > c > d$
2. $b > c > d > a$
3. $c > d > a > b$
4. $a > b > c > d$
5. $b > c > d > a$
6. $c > d > a > b$
7. $a > b > c > d$

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1. c gets 20 points
2. b gets 19 points
3. a gets 18 points
4. d gets 13 points

Borda Example

Let's get rid of d.

1. $a > b > c > d$
2. $b > c > d > a$
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Borda Example

Let's get rid of d.

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2. $b > c > a$
3. $c > a > b$
4. $a > b > c$
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3. $c > a > b$
4. $a > b > c$
5. $b > c > a$
6. $c > a > b$
7. $a > b > c$

1. a gets 15 points
2. b gets 14 points
3. c gets 13 points

Voting Summary

1. Use Borda count whenever possible.
2. Practically, Borda requires calculating all preferences: often computationally hard.

Painting the House

Name	Wants house painted?
Alice	Yes
Bob	No
Caroline	Yes
Donald	Yes
Emily	Yes

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- Each agent i has a type $\theta_i \in \Theta_i$ which is private.

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- The protocol results in some outcome $o \in O$.
- Each agent i gets a value $v_i(o, \theta_i)$ for outcome o.
- The social choice function $f(\theta)$ tells us the outcome we want to achieve. For example,

$$f(\theta) = \arg\max_{o \in O} \sum_{i=1}^{n} v_i(o, \theta_i) \tag{1}$$

Painting the House

Name	Type (θ_i)	$v_i(Paint, \theta_i)$	$v_i(NoPaint, \theta_i)$
Alice	WantPaint	10	0
Bob	DontNeedPaint	0	0
Caroline	WantPaint	10	0
Donald	WantPaint	10	0
Emily	WantPaint	10	0

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- Try this: Everyone votes Y/N. If majority votes Y then paint house. All pay 1/5 of cost (4 each).

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- Try this: Everyone votes Y/N. If majority votes Y then paint house. All pay 1/5 of cost (4 each).
- Bob must pay for a paint job he didn't want.

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- Try this: Everyone votes Y/N. Split cost among those who voted Y.
- There is an incentive for all but Bob to lie.

Definition (g Implements f)

A mechanism $g : S_1 \times \cdots \times S_A \rightarrow O$ implements social choice function $f(\cdot)$ if there is an equilibrium strategy profile $(S_1^*(\cdot), \ldots, S_A^*(\cdot))$ of the game induced by g such that $g(S_1^*(\theta_1), \ldots, S_A^*(\theta_A)) = f(\theta_1, \ldots, \theta_A)$ for all $\theta \in \Theta$.

Where we let $S_i(\theta_i)$ be agent i's strategy given that it is of type θ_i.

Definition (Dominant Strategy Equilibrium)

A strategy profile $(S_1^*(\cdot), \ldots, S_A^*(\cdot))$ of the game induced by g is a dominant strategy equilibrium if for all i and all θ_i,

$$v_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq v_i(g(s_i', s_{-i}), \theta_i)$$

for all $s_i' \in S_i$ and all $s_{-i} \in S_{-i}$.

Definition (g Implements f)

A mechanism $g : S_1 \times \cdots \times S_A \to O$ implements social choice function $f(\cdot)$ in dominant strategies if there is a dominant strategy equilibrium strategy profile $(S_1^*(\cdot), \ldots, S_A^*(\cdot))$ of the game induced by g such that $g(S_1^*(\theta_1), \ldots, S_A^*(\theta_A)) = f(\theta_1, \ldots, \theta_A)$ for all $\theta \in \Theta$.

Definition (Strategy-Proof)

The social choice function $f(\cdot)$ is truthfully implementable in dominant strategies (or strategy-proof) if $s_i^*(\theta_i) = \theta_i$ (for all $\theta_i \in \Theta_i$ and all i) is a dominant strategy equilibrium of the direct revelation mechanism $f(\cdot)$. That is, if for all i and all $\theta_i \in \Theta_i$,

$$v_i(f(\theta_i, \theta_{-i}), \theta_i) \geq v_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)$$

for all $\hat{\theta}_i \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.

Theorem (Revelation Principle)

If there exists a mechanism g that implements the social choice function f in dominant strategies then f is truthfully implementable in dominant strategies.

Selling Example

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Set $p(o)$ such that the agent who wins must pay a tax equal to the second highest valuation. No one else pays/gets anything.

$$u_i(o, \theta_i) = \begin{cases} \theta_i - \max_{j \neq i} \theta_j & \text{if } o = i \\ 0 & \text{otherwise.} \end{cases}$$

Truth-Telling is Dominant in Vickrey Payments Example.

1. Let $b_i(\theta_i)$ be i's bid given that his true valuation is θ_i.

□

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$$u_i(i, \theta_i) = \theta_i - b' > 0$$

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4. If $b' > \theta_i$ then any bid $b_i(\theta_i) < b'$ is optimal since

$$u_i(i, \theta_i) = 0$$

□

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5. Since we have that if $b' < \theta_i$ then i should bid $> b'$ and if $b' > \theta_i$ then i should bid $< b'$, and we don't know b' then i should bid θ_i.

□

Theorem (Groves-Clarke Mechanism)

If

$$f(\theta) = \arg\max_{o \in O} \sum_{i=1}^{n} v_i(o, \theta_i)$$

then calculating the outcome using

$$f(\tilde{\theta}) = \arg\max_{o \in O} \sum_{i=1}^{n} v_i(o, \tilde{\theta}_i)$$

(where $\tilde{\theta}$ *are reported types) and giving the agents payments of*

$$p_i(\tilde{\theta}) = \sum_{j \neq i} v_j(f(\tilde{\theta}), \tilde{\theta}_j) - h_i(\tilde{\theta}_{-i})$$

(where $h_i(\theta_{-i})$ *is an arbitrary function) results in a strategy-proof mechanism.*

Groves-Clarke Payments for House Painting

Name	$v_i(o,\tilde{\theta})$	$v_i(o,\theta) + \sum_{j \neq i} v_j(\tilde{\theta})$
Alice	$10 - \frac{20}{4} = 5$	
Bob	$0 - 0 = 0$	
Caroline	$10 - \frac{20}{4} = 5$	
Donald	$10 - \frac{20}{4} = 5$	
Emily	$10 - \frac{20}{4} = 5$	

Assuming all tell the truth.

Groves-Clarke Payments for House Painting

Name	$v_i(o, \tilde{\theta})$	$v_i(o, \theta) + \sum_{j \neq i} v_j(\tilde{\theta})$
Alice	$10 - \frac{20}{4} = 5$	$5 + 15 = 20$
Bob	$0 - 0 = 0$	$0 + 20 = 20$
Caroline	$10 - \frac{20}{4} = 5$	$5 + 15 = 20$
Donald	$10 - \frac{20}{4} = 5$	$5 + 15 = 20$
Emily	$10 - \frac{20}{4} = 5$	$5 + 15 = 20$

Assuming all tell the truth.

Groves-Clarke Payments for House Painting

Name	$v_i(o, \tilde{\theta})$	$v_i(o, \theta) + \sum_{j \neq i} v_j(\tilde{\theta})$
Alice	$0 - 0 = 0$	$10 + (\frac{10}{3} \cdot 3) = 20$
Bob	$0 - 0 = 0$	$0 + (\frac{10}{3} \cdot 3) = 10$
Caroline	$10 - \frac{20}{3} = \frac{10}{3}$	$10 + (\frac{10}{3} \cdot 2) = 10$
Donald	$10 - \frac{20}{3} = \frac{10}{3}$	$10 + (\frac{10}{3} \cdot 2) = 10$
Emily	$10 - \frac{20}{3} = \frac{10}{3}$	$10 + (\frac{10}{3} \cdot 2) = 10$

Alice lies.

Theorem (Vickrey-Clarke-Groves (VCG) Mechanism)

If

$$f(\theta) = \arg\max_{o \in O} \sum_{i=1}^{n} v_i(o, \theta_i)$$

then calculating the outcome using

$$f(\tilde{\theta}) = \arg\max_{o \in O} \sum_{i=1}^{n} v_i(o, \tilde{\theta}_i)$$

(where $\tilde{\theta}$ are reported types) and giving the agents payments of

$$p_i(\tilde{\theta}) = \sum_{j \neq i} v_j(f(\tilde{\theta}_{-i}), \tilde{\theta}_j) - \sum_{j \neq i} v_j(f(\tilde{\theta}), \tilde{\theta}_j)$$

(where $h_i(\theta_{-i})$ is an arbitrary function) results in a strategy-proof mechanism.

VCG Payments for House Painting

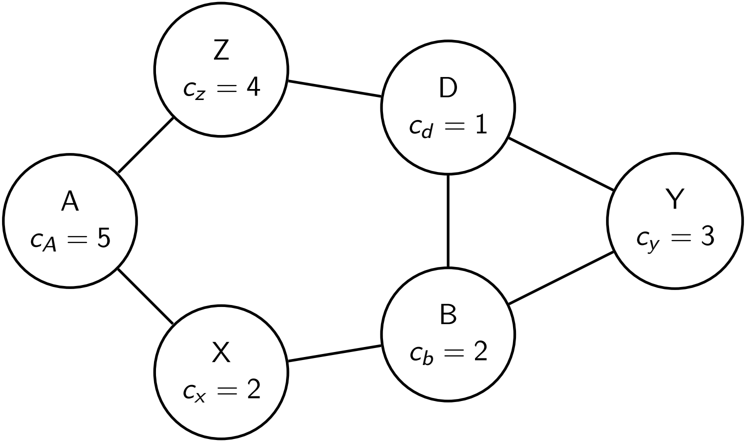
Name	$v_i(o, \tilde{\theta})$	$\sum_{j \neq i} v_j(f(\tilde{\theta}_{-i}), \tilde{\theta}_j)$	$\sum_{j \neq i} v_j(\cdot) - \sum_{j \neq i} v_j(\cdot)$
Alice	$10 - \frac{20}{4} = 5$	$(10 - \frac{20}{3}) \cdot 3 = 10$	$10 - 15 = 5$
Bob	$0 - 0 = 0$	$(10 - \frac{20}{4}) \cdot 4 = 20$	$20 - 20 = 0$
Caroline	$10 - \frac{20}{4} = 5$	$(10 - \frac{20}{3}) \cdot 3 = 10$	$10 - 15 = 5$
Donald	$10 - \frac{20}{4} = 5$	$(10 - \frac{20}{3}) \cdot 3 = 10$	$10 - 15 = 5$
Emily	$10 - \frac{20}{4} = 5$	$(10 - \frac{20}{3}) \cdot 3 = 10$	$10 - 15 = 5$

Alice tells the truth.

Distributed Algorithmic Mechanism Design

- Algorithmic mechanism design: make mechanism polynomial time
- Distributed algorithmic mechanism design: make mechanism distributed

Inter-Domain Routing Problem



Conclusion

- GC and VCG payment equations are useful ready-made solutions to many problems.

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- But, we still need more research into how to distribute the mechanisms and how to make their calculation computationally tractable.