

Coalition Formation

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Abstract

We present the coalition formation problem and some solutions (Sandholm et al., 1999; Shehory and Kraus, 1998).



Characteristic Form Games

- $A = \{1, \dots, A\}$ is the set of agents,
- $\mathbf{u} = (u_1, \dots, u_A) \in \mathfrak{R}^A$ is the **outcome**
- $V(\cdot)$ is a **rule** that maps every coalition $S \subset A$ to a utility possibility set: $V(S) \subset \mathfrak{R}^S$.



Transferable Utility Game

- $A = \{1, \dots, A\}$ is the set of agents,
- $v(\cdot)$ is a **characteristic** function that gives every coalition $S \subset A$ a worth $v(S) \in \mathfrak{R}$.

In both games we want to maximize the worth/utility.



Sample Problems

- task allocation problem (let tasks be the agents),
- sensor network problems (agents must form groups),
- distributed winner determination in combinatorial auctions,
- agents grouping to handle workflows (just-in-time incorporation).



Example

S	$v(S)$
(1)	2
(2)	2
(3)	4
(12)	5
(13)	7
(23)	8
(123)	9

$$(1)(2)(3)$$

$$2 + 2 + 4 = 8$$

$$(1)(23)$$

$$2 + 8 = 10$$

$$(2)(13)$$

$$2 + 7 = 9$$

$$(3)(12)$$

$$4 + 5 = 9$$

$$(123)$$

$$9$$



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$\mathbf{u} = \{5, 5, 5\}$, is that feasible?



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$\mathbf{u} = \{5, 5, 5\}$, is that feasible? No



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$\mathbf{u} = \{2, 2, 2\}$, is that feasible?



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$\mathbf{u} = \{2, 2, 2\}$, is that feasible? Yes, but it is not stable.



Definition (Core)

An outcome \mathbf{u} is in the **core** if

1

$$\forall S \subset A : \sum_{i \in S} \mathbf{u}_i \geq v(S)$$

2 it is feasible.

Where, in superadditive domains feasibility corresponds to having

$$\sum_{i \in A} \mathbf{u}_i = v(A)$$



Example

S	$v(S)$
(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

(1)(2)(3)

$$1 + 2 + 2 = 5$$

(1)(23)

$$1 + 4 = 5$$

(2)(13)

$$2 + 3 = 5$$

(3)(12)

$$2 + 4 = 6$$

u	in Core?
{2, 2, 2}	
{2, 2, 3}	
{1, 2, 2}	

(123)

6



Example

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(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

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$$1 + 2 + 2 = 5$$

$$(1)(23)$$

$$1 + 4 = 5$$

$$(2)(13)$$

$$2 + 3 = 5$$

$$(3)(12)$$

$$2 + 4 = 6$$

u	in Core?
$\{2, 2, 2\}$	yes
$\{2, 2, 3\}$	
$\{1, 2, 2\}$	

$$(123)$$

$$6$$



Example

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(1)	1
(2)	2
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$$(1)(2)(3)$$

$$1 + 2 + 2 = 5$$

$$(1)(23)$$

$$1 + 4 = 5$$

$$(2)(13)$$

$$2 + 3 = 5$$

$$(3)(12)$$

$$2 + 4 = 6$$

u	in Core?
$\{2, 2, 2\}$	yes
$\{2, 2, 3\}$	no
$\{1, 2, 2\}$	

$$(123)$$

$$6$$



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(1)	1
(2)	2
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$$(1)(2)(3)$$

$$1 + 2 + 2 = 5$$

$$(1)(23)$$

$$1 + 4 = 5$$

$$(2)(13)$$

$$2 + 3 = 5$$

$$(3)(12)$$

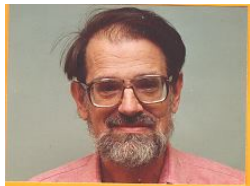
$$2 + 4 = 6$$

u	in Core?
$\{2, 2, 2\}$	yes
$\{2, 2, 3\}$	no
$\{1, 2, 2\}$	no

$$(123)$$

$$6$$

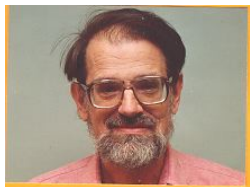




Lloyd Shapley

- How do we find an appropriate outcome?
- How do we **fairly** distribute the outcomes' value?
- What is fair?





Lloyd Shapley

- How do we find an appropriate outcome?
- How do we **fairly** distribute the outcomes' value?
- What is fair?

The Shapley value gives us one specific set of payments for coalition members, which are deemed fair.



Example

S	$v(S)$
$()$	0
(1)	1
(2)	3
(12)	6



Definition (Shapley Value)

Let $B(\pi, i)$ be the set of agents in ordering π that come before agent i . The **Shapley value** for agent i given A agents is given by

$$Sh(A, i) = \frac{1}{A!} \sum_{\pi} v(B(\pi, i) \cup i) - v(B(\pi, i)),$$

where the sum is over all possible orderings of the agents.



Example

$$\begin{aligned} Sh(\{1,2\},1) &= \frac{1}{2} \cdot (v(1) - v() + v(21) - v(2)) \\ &= \frac{1}{2} \cdot (1 - 0 + 6 - 3) = 2 \\ Sh(\{1,2\},2) &= \frac{1}{2} \cdot (v(12) - v(1) + v(2) - v()) \\ &= \frac{1}{2} \cdot (6 - 1 + 3 - 0) = 4 \end{aligned}$$

Note that the Shapley outcome is always feasible.



Drawbacks

- Requires calculating $A!$ orderings.
- Requires knowing $v(\cdot)$ for all coalitions.
- We still need to find the coalition structure.



Brute Force Search

(1)(2)(3)(4)

(12)(3)(4) (13)(2)(4) (14)(2)(3) (23)(1)(4) (24)(1)(3) (34)(1)(2)

(1)(234) (2)(134) (3)(124) (4)(123) (12)(34) (14)(23) (13)(24)

(1234)



Brute Force Search

(1)(2)(3)(4)

(12)(3)(4) (13)(2)(4) (14)(2)(3) (23)(1)(4) (24)(1)(3) (34)(1)(2)

(1)(234) (2)(134) (3)(124) (4)(123) (12)(34) (14)(23) (13)(24)

(1234)

All possible coalitions



Search Order Bounds

Level	Bound
A	$A/2$
$A-1$	$A/2$
$A-2$	$A/3$
$A-3$	$A/3$
$A-4$	$A/4$
$A-5$	$A/4$
:	:
2	A
1	none



FIND-COALITION(i)

- 1 $L_i \leftarrow$ set of all coalitions that include i .
- 2 $S_i^* \leftarrow \arg \max_{S \in L_i} v_i(S)$
- 3 $w_i^* \leftarrow v_i(S_i^*)$
- 4 Broadcast (w_i^*, S_i^*) and wait for all other broadcasts.
Put into W^*, S^* sets.
- 5 $w_{\max} = \max W^*$ and S_{\max} is the corresponding coalition.
- 6 **if** $i \in S_{\max}$
- 7 **then** join S_{\max}
- 8 Delete S_{\max} from L_i .
- 9 Delete all $S \in L_i$ which include agents from S_{\max} .
- 10 **if** L_i is not empty
- 11 **then** goto 2
- 12 **return**



Sandholm, T., Larson, K., Anderson, M., Shehory, O., and Tohmé, F. (1999).

Coalition structure generation with worst case guarantees.

Artificial Intelligence, 111(1-2):209–238.

Shehory, O. and Kraus, S. (1998).

Methods for task allocation via agent coalition formation.

Artificial Intelligence, 101(1-2):165–200.

