Coalition Formation

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September 29, 2005

Abstract

We present the coalition formation problem and some solutions (Sandholm et al., 1999; Shehory and Kraus, 1998).

Characteristic Form Games

- $A = \{1, \ldots, A\}$ is the set of agents,
- $\mathbf{u} = (u_1, \ldots, u_A) \in \Re^A$ is the outcome
- $V(\cdot)$ is a rule that maps every coalition $S \subset A$ to a utility possibility set: $V(S) \subset \Re^S$.

Transferable Utility Game

- $A = \{1, \ldots, A\}$ is the set of agents,
- $v(\cdot)$ is a characteristic function that gives every coalition $S \subset A$ a worth $v(S) \in \Re$.

In both games we want to maximize the worth/utility.

Sample Problems

- task allocation problem (let tasks be the agents),
- sensor network problems (agents must form groups),
- distributed winner determination in combinatorial auctions,
- agents grouping to handle workflows (just-in-time incorporation).

Example

S	$v(S)$
(1)	2
(2)	2
(3)	4
(12)	5
(13)	7
(23)	8
(123)	9

(1)(2)(3)
$2+2+4=8$

(1)(23)
$2+8=10$

(2)(13)
$2+7=9$

(3)(12)
$4+5=9$

(123)
9

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$\mathbf{u}=\{5,5,5\}$, is that feasible?

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$\mathbf{u}=\{5,5,5\}$, is that feasible? No

Example

S	$v(S)$
(1)	2
(2)	2
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(12)	5
(13)	7
(23)	8
(123)	9

(1)(2)(3)
$2+2+4=8$

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(2)(13)
$2+7=9$

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(123)
9

$\mathbf{u} = \{2, 2, 2\}$, is that feasible?

Example

S	$v(S)$
(1)	2
(2)	2
(3)	4
(12)	5
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$\mathbf{u} = \{2,2,2\}$, is that feasible? Yes, but it is not stable.

Definition (Core)

An outcome **u** is in the core if

1. $$\forall_{S \subset A} : \sum_{i \in S} \mathbf{u}_i \geq v(S)$$
2. it is feasible.

Where, in superadditve domains feasibility corresponds to having

$$\sum_{i \in A} \mathbf{u}_i = v(A)$$

Example

S	$v(S)$
(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

(1)(2)(3)
$1+2+2=5$

(1)(23)
$1+4=5$

(2)(13)
$2+3=5$

(3)(12)
$2+4=6$

(123)
6

u	in Core?
$\{2,2,2\}$	
$\{2,2,3\}$	
$\{1,2,2\}$	

Example

S	$v(S)$
(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

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$1+2+2=5$

(1)(23)
$1+4=5$

(2)(13)
$2+3=5$

(3)(12)
$2+4=6$

u	in Core?
$\{2,2,2\}$	yes
$\{2,2,3\}$	
$\{1,2,2\}$	

(123)
6

Example

S	$v(S)$
(1)	1
(2)	2
(3)	2
(12)	4
(13)	3
(23)	4
(123)	6

(1)(2)(3)
$1+2+2=5$

(1)(23)
$1+4=5$

(2)(13)
$2+3=5$

(3)(12)
$2+4=6$

u	in Core?
$\{2,2,2\}$	yes
$\{2,2,3\}$	no
$\{1,2,2\}$	

(123)
6

Example

S	$v(S)$
(1)	1
(2)	2
(3)	2
(12)	4
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(1)(23)
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(2)(13)
$2+3=5$

(3)(12)
$2+4=6$

u	in Core?
$\{2,2,2\}$	yes
$\{2,2,3\}$	no
$\{1,2,2\}$	no

(123)
6

Lloyd Shapley

- How do we find an appropiate outcome?
- How do we fairly distribute the outcomes' value?
- What is fair?

Lloyd Shapley

- How do we find an appropiate outcome?
- How do we fairly distribute the outcomes' value?
- What is fair?

The Shapley value gives us one specific set of payments for coalition members, which are deemed fair.

Example

S	$v(S)$
()	0
(1)	1
(2)	3
(12)	6

Definition (Shapley Value)

Let $B(\pi, i)$ be the set of agents in ordering π that come before agent i. The Shapley value for agent i given A agents is given by

$$Sh(A,i) = \frac{1}{A!} \sum_{\pi} v(B(\pi,i) \cup i) - v(B(\pi,i)),$$

where the sum is over all possible orderings of the agents.

Example

$$
\begin{aligned}
Sh(\{1,2\},1) &= \frac{1}{2}\cdot(v(1)-v()+v(21)-v(2)) \\
&= \frac{1}{2}\cdot(1-0+6-3)=2 \\
Sh(\{1,2\},2) &= \frac{1}{2}\cdot(v(12)-v(1)+v(2)-v()) \\
&= \frac{1}{2}\cdot(6-1+3-0)=4
\end{aligned}
$$

Note that the Shapley outcome is always feasible.

Drawbacks

- Requires calculating $A!$ orderings.
- Requires knowning $v(\cdot)$ for all coaltions.
- We still need to find the coalition structure.

Brute Force Search

(1)(2)(3)(4)

(12)(3)(4) (13)(2)(4) (14)(2)(3) (23)(1)(4) (24)(1)(3) (34)(1)(2)

(1)(234) (2)(134) (3)(124) (4)(123) (12)(34) (14)(23) (13)(24)

(1234)

Brute Force Search

(1)(2)(3)(4)

(12)(3)(4) (13)(2)(4) (14)(2)(3) (23)(1)(4) (24)(1)(3) (34)(1)(2)

(1)(234) (2)(134) (3)(124) (4)(123) (12)(34) (14)(23) (13)(24)

(1234)

All possible coalitions

Search Order Bounds

Level	Bound
A	$A/2$
$A-1$	$A/2$
$A-2$	$A/3$
$A-3$	$A/3$
$A-4$	$A/4$
$A-5$	$A/4$
$\vdots$	$\vdots$
2	A
1	none

FIND-COALITION(i)

```
1   L_i ← set of all coalitions that include i.
2   S_i* ← arg max_{S∈L_i} v_i(S)
3   w_i* ← v_i(S_i*)
4   Broadcast (w_i*, S_i*) and wait for all other broadcasts.
    Put into W*, S* sets.
5   w_max = max W* and S_max is the corresponding coalition.
6   if i ∈ S_max
7       then join S_max
8   Delete S_max from L_i.
9   Delete all S ∈ L_i which include agents from S_max.
10  if L_i is not empty
11      then goto 2
12  return
```

Sandholm, T., Larson, K., Anderson, M., Shehory, O., and Tohmé, F. (1999).
Coalition structure generation with worst case guarantees.
Artificial Intelligence, 111(1-2):209–238.

Shehory, O. and Kraus, S. (1998).
Methods for task allocation via agent coalition formation.
Artificial Intelligence, 101(1-2):165–200.